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New methods of the mass and heat transfer theory—I. The method of asymptotic correction and the method of model equations and analogies

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Abstract—Two new approximate analytical methods are suggested to investigate the heat and mass transfer problems—the method of asymptotic correction and the method of model equations and analogies.

A number of specific examples are considered illustrating the use of the above methods. New formulae have been derived which are of interest for their own sake in the convective mass and heat transfer theory.

Comparison of the results obtained with a variety of special typical cases for whose checking exact, approximate or asymptotic formulae are already available shows a good accuracy and wide possibilities of the methods suggested. These methods can also be successfully used to construct approximate (engineering) formulae in the problems of chemical engineering, chemical mechanics, microkinetics, hydrodynamics, etc.

1. INTRODUCTION

IN THE OVERWHELMING majority of convective mass and heat transfer problems exact analytical solutions are unobtainable. The main reasons for this are, as a rule, due to the non-linear nature of equations or boundary conditions, to the dependence of equation coefficients on the coordinates, to complex shapes of the boundaries, etc. Moreover, even in those cases when an exact solution to some problem has been found in an explicit analytical form, it may turn to be inconvenient and sometimes even useless for physical and mathematical interpretations or numerical calculations. Therefore in order to obtain the required information about the phenomenon or the process investigated, one has to resort to different kinds of simplifications in the mathematical statement of the problem and also to different approximations, numerical methods or to combinations of both simultaneously.

Among the approximate analytical methods, the most fruitful and general ones are the methods of perturbations (asymptotic expansions) in large and small values of some parameter or coordinate [1–3]. The resulting asymptotic series often diverge or very slowly converge. Moreover, it is generally possible to calculate only several (usually no more than two or three) first terms of perturbated expansion. These circumstances preclude the assessment of the solution behaviour at intermediate (finite) values of the parameter or coordinate and impose rigorous restrictions on the use of asymptotic formulae in engineering practice.

There were attempts to improve the asymptotic series by using the Shenks and Euler transformations, making approximations by rational fractions, choosing natural (optimal) coordinates, linking coordinate or parametric expansions etc (see, e.g. [1] and [4]). Despite being useful, all of these transformations and approaches do not unfortunately possess generality and have the drawback that one has to use them intuitively, without understanding the mechanism involved. Moreover, that all of the above methods could be used, it is necessary that a fairly great number of terms in the initial asymptotic expansion be available (which, as a rule, are now known in advance and whose derivation usually involves extensive difficulties).

Despite the above drawbacks, the perturbation methods are far from becoming less important at the present time due to rapid advances in computer technology. They afford one of the most powerful tools of the present-day applied mathematics and serve to elucidate the qualitative aspects of very complex linear and non-linear boundary-value problems, to obtain the asymptotic expressions and analyse the solutions in the vicinity of singular points to construct the reference 'test' solutions, and in a number of cases they form a basis for developing the computational methods.

The inconvenience of directly using the results of asymptotic analysis for practical purpose can be largely obviated by applying the computer-based calculation methods. The numerical methods possess great universality and allow one to efficiently obtain the solutions of different kinds of problems for the intermediate values of the parameter and coordinate (i.e. in the region where the asymptotic methods fail). However, despite the obvious usefulness and generality of the computational methods their inportance is sometimes unjustifiably overemphasized. Specifically, among the drawbacks of the numerical methods are: first, the absence of an approximate analytical form of the solution (approximate formulae are often more convenient than tables and graphs; moreover, a large number and a wide range of the characteristic

NOMENCLATURE

a droplet or solid particle radius; characteristic particle dimension

C dimensionless concentration in the problem with volumetric chemical reaction, C_*/C_s

C_* concentration

C_s concentration on particle surface

C_∞ unperturbed concentration at infinity

C_{m*} mth component concentration

$C_{m\infty}$ mth component concentration at infinity

C_x resistance coefficient

c dimensionless concentration, $\dfrac{C_\infty - C_*}{C_\infty}$

c_m dimensionless concentration of mth component, $\dfrac{C_{m\infty} - C_{m*}}{C_{m\infty}}$

D diffusion coefficient

D_m mth component diffusion coefficient

$\mathbf{F}$ total body resistance force

$F_s(F_{ms})$ function determining the kinetics of surface chemical reaction (for mth component)

F_v function determining the kinetics of volumetric chemical reaction

$f_s(c)$ $= \dfrac{F_s(C_*)}{F_s(C_\infty)}$

$f_v(C)$ $= \dfrac{F_v(C_*)}{F_v(C_s)}$

G shear coefficient

H_m reaction heat of mth component

I dimensionless total (integral) diffusional flux per particle

$\mathbf{i}$ unit directing liquid velocity vector at infinity

j dimensionless local diffusional flux

K_s surface chemical reaction rate constant

K_v volumetric chemical reaction rate constant

k, p formal parameters of the problem [see equation (1)]

k_s dimensionless surface reaction rate constant, $aK_sF_s(C_\infty)(DC_\infty)^{-1}$

k_v dimensionless volumetric chemical reaction rate constant, $a^2K_vF_v(C_s)(DC_s)^{-1}$

M number of species participating in reaction

Nu average Nusselt number (based on particle radius)

n order of volumetric or surface chemical reaction

Pe diffusional Peclet number, aUD^{-1}

Pe_T thermal Peclet number, $aU\chi^{-1}$

Pe_m diffusional Peclet number of mth component, aUD_m^{-1}

Q similarity parameter, $Sh_\infty/Sh_\beta = \text{const.} \times Pe^{-1/6}(\beta+1)^{1/2}$

Re Reynolds number, aUv^{-1}

r_* radial coordinate

r dimensionless radial coordinate, r_*/a

r, θ, φ spherical coordinate system with origin fixed at droplet or particle center

S sought-after quantity

$S_{k\infty}, S_{\infty p}$ exact asymptotic expressions of the sought-after quantity S for $p \to \infty$ and $k \to \infty$, respectively

$S^*_{k\infty}, S^*_{\infty p}$ asymptotic expressions of initial approximate equation for S when $p \to \infty$ and $k \to \infty$

Sh mean Sherwood number based on particle radius [equations (25) and (50)]

Sh_0 mean Sherwood number corresponding to purely diffusional mode of reaction on particle surface

Sh_β mean Sherwood number obtained in diffusional boundary layer approximation for a moderately viscous droplet [for translational Stokes flow it is specified by formula (7), for shear flow, by equation (74)]

Sh_∞ mean Sherwood number obtained in diffusional boundary layer approximation for a solid sphere [for translational Stokes flow it is specified by formula (7), for shear flow, by equation (74)]

Sh_0 mean Sherwood number corresponding to developed mode of diffusion for $t \to \infty$, $\lim\limits_{t\to\infty} Sh$

Sh_m mean Sherwood number for mth component of mixture

s analog of quantity S

T_* temperature in flow

T_∞ unperturbed temperature at infinity

t_* time

t dimensionless time, $a^{-2}Dt_*$

U characteristic flow velocity: for translational flow $U = U_\infty$, for shear flow $U = aG$

U_∞ unperturbed flow velocity at infinity

$\mathbf{V}$ liquid velocity vector

$\mathbf{v}$ dimensionless liquid velocity vector, $\mathbf{V}/U$

X_1, X_2, X_3 rectangular Cartesian coordinate system fixed in particle; $x_i = X_i/a$ $(i = 1, 2, 3)$, $x \equiv x_1$

y $= r - 1$.

Greek symbols

β	ratio of dynamic viscosities of droplet and surrounding liquid ($\beta = 0$ corresponds to gas bubble; $\beta = \infty$, to solid particle)
Γ	particle surface
Δ	Laplace operator
θ	spherical angle reckoned from forward stagnation point of particle
λ	thermal conductivity of liquid
λ_1, λ_2	coefficients of Hadamard–Rybczinskii stream function expansion near droplet surface, equation (64)
ξ	$= x - 1$
ρ	liquid density
Σ_*, Σ	dimensional and nondimensional surface area of particle, respectively
χ	thermal diffusivity of liquid
ψ	Hadamard–Rybczinskii stream function, formula (65)
ω	angle between body rotation axis and free stream direction.

parameters of a problem generally make the results of numerical calculations obscure and inconvenient for interpretation and practical application); second, considerable numerical effort and large time expenditures as compared with approximate analytical methods; third, the necessity of resorting all the same to asymptotic methods in order to obtain the desired quantities in the regions with large solution gradients and also to investigate the problems in geometrically extended regions and in the regions with a non-homogeneous microstructure; fourth, the lack of 'local' universality of the computational methods, i.e. the change in the region geometry, flow pattern, reaction kinetics, etc. in analogous problems each time requires changes, and sometimes substantial, in the computer programmes (analytical results often allow us straightway to take into account the above possible modifications of the problem).

Up till now, different, often scarcely justified and in many respects relying on purely intuitive grounds, approximate engineering methods still retain their importance, among which there are, for example: one-parametric integral methods in the theory of laminar and turbulent boundary layers; the method of equally accessible surface; film and penetration models in the problems of mass transfer with chemical reactions; different modifications of the method of linearization of equations or boundary conditions, etc. For many cases of practical interest the use of these simple methods turns out to be quite sufficient. The general drawback of the engineering-type methods is their level of inaccuracy. Moreover, these methods are based, as a rule, on specific physical concepts of the mechanism underlying the process considered and therefore are not quite general.

It is clear from the above that in order to obtain the necessary information on actual phenomena and processes, it is necessary that new non-traditional approaches and analytical methods for the solution of corresponding problems be developed in addition to the well-known classical methods of investigation. Great attention should also be paid to the principles and means of constructing approximate engineering formulae which would be convenient for practical application. Simplicity, accuracy, generality and informativeness are the basic requirements which should be met by approximate methods and formulae. The main objective of the authors of this paper was to describe and illustrate new approximate methods of problem investigations and principles of engineering formulae construction, which meet, to a certain extent, the above requirements and ultimately allow one to obtain the required relations in simple analytical form (in the form of explicit expressions or algebraic equations). The essential feature of such approximate formulae is their high informativeness which allows the same formula to be used for the description of a variety of problems, phenomena or processes similar in type but differing in shape and form of the reacting surface (solid or liquid), flow geometry, etc. Most of the methods suggested are based on a universal procedure of the asymptotic correction of approximate relations, which is presented below.

2. THE METHOD OF ASYMPTOTIC CORRECTION

It is common to use different engineering formulae that were obtained empirically or by means of approximate solution of corresponding (boundary-value) problems. The region of applicability of this kind of formulae is generally limited and should be specified in each individual case. Below, a simple universal method for effective improvement of approximate engineering formulae is suggested, which is based on the use of exact asymptotic expressions of the initial boundary-value problem.

Let for the basic required quantity S the following expression be obtained

$$S = S(k, p), \tag{1}$$

which correctly reflects the qualitative behaviour of S depending on a change in the governing parameters of the problem, k and p (here and hereafter it is assumed for simplicity that there are two such parameters and that $0 \leqslant k, p < \infty$). Let the main terms of the approximations for the asymptotes to expression (1) in the limiting

cases $k \to \infty$ ($p = \text{const.}$) and $p \to \infty$ ($k = \text{const.}$) are

$$k \to \infty, S \to S^*_{\infty,p}; \quad p \to \infty, S \to S^*_{k\infty} \tag{2}$$

$$S^*_{\infty,p} = S^*_{\infty,p}(k,p), \quad S^*_{k\infty} = S^*_{k\infty}(k,p) \tag{3}$$

[instead of equation (2), any other limiting cases can be considered; see below for specific examples].

If similar exact asymptotic expressions of the solution for the initial boundary-value problem are known

$$k \to \infty, S \to S_{\infty p}; \quad p \to \infty, S \to S_{k\infty}, \tag{4}$$

then approximate formula (1) can be much improved in the following simple way. Eliminate the parameters k and p from equation (3) through the use of $S^*_{\infty,p}$ and $S^*_{k\infty}$ (the corresponding transformation is assumed to be non-degenerate) and substitute the latter into formula (1). This will give

$$S = \Phi(S^*_{\infty p}, S^*_{k\infty}) \tag{5}$$

$$[\Phi(S^*_{\infty p}(k,p), S^*_{k\infty}(k,p)) \equiv S(k,p)]$$

Placing now, into this expression, relations (4) instead of equation (3) will yield the following formula

$$S = \Phi(S_{\infty p}, S_{k\infty}), \tag{6}$$

which, apart from a correct qualitative description of S, now provides an exact result in the limiting cases $k \to \infty$ and $p \to \infty$ [in contrast to equation (1)].

It should be noted that if formula (1) is an exact one, then formula (6) will also be exact (i.e. the procedure suggested does not obviously deteriorate the accuracy of the initial problem). Moreover, it is easy to verify that if equation (1) will differ, due to arbitrary stretching of the parameters k and p ($k \to \alpha k$, $p \to \beta p$; $\alpha, \beta = \text{const.}$), from an exact one [i.e. if equation (1) is 'spoilt' by stretching], then the above procedure with the use of the known exact asymptotic expressions restores it completely and makes it exact. This is a consequence of the fact that the stretching of the parameters in the initial equation (1) leads to a corresponding (exactly the same) stretching of these parameters in asymptotic expressions (2) and (3), by virtue of which the elimination of the stretched parameters through the use of $S^*_{\infty\beta p}$ and $S^*_{\alpha k\infty}$ leads to the same expression (5). It should be added that apart from this there is also a wide class of transformations after which the proposed procedure also restores the 'spoilt' formula (for example, the transformation of the form $k \to \alpha k^n$, $p \to \beta p^m$; $n, m > 0$).

The above procedure of asymptotic correction is extremely simple and convenient. It can be used with success for improving various kinds of engineering formulae obtained both by approximate analytical and numerical methods and also empirically, i.e. by processing experimental data. It should also be stressed that improved expression (6) can generally be used for the approximate description of a much wider range of phenomena or processes than the initial formula (1) [in this sense it can be said that the formulae of type (6) are more informative]. This latter circumstance is due to the fact that the functional relationship represented by equation (6) remains the same for a rather wide class of similar problems, while specific modifications and differences (the type of flow, shape of a particle, etc.) of these problems are taken into account only by corresponding correcting asymptotic parameters $S_{\infty p}$ and $S_{k\infty}$.

Note also that initial expression (1) and final result (6) can also be written in a non-explicit form as an algebraic equation for the quantity S [see, for example, formulae (9) and (12)]; moreover, in a number of cases a similar procedure can be successfully used to improve the coefficients of approximate differential equations which have been obtained from the initial more complex equations by applying various simplifications.

It should also be stressed that expressions (3) and (4) may involve at a time several main terms of the corresponding asymptotic expansions.

The proposed method can be illustrated on several specific examples.

It is known [5] that in analyzing the mass transfer between a droplet and a translational Stokes flow [the distribution of liquid velocities outside the droplet is governed by the stream function (65)] in the diffusional boundary layer approximation, the analytical solution of the problem [the formulation of which is given later, equations (62)–(64)] can be obtained only in the limiting cases of relatively small and infinitely large values of the droplet and surrounding liquid viscosity ratio, β. This is because the asymptotic expansions for concentration and mean Sherwood number in large Peclet number $Pe = aU_\infty D^{-1}$ (a is the droplet radius, U_∞ the unperturbed flow velocity at infinity, D the diffusion coefficient) for a droplet are not uniformly applicable in the parameter β. Thus, the asymptotic values for the mean Sherwood number at $Pe \gg 1$ for a moderately viscous droplet [$\beta = 0(1)$] and a solid sphere ($\beta = \infty$) are [5]

$$\beta = 0(1),\ Sh = Sh_\beta; \quad \beta \to \infty,\ Sh \to Sh_\infty \tag{7}$$

$$Sh_\beta = \left[\frac{2Pe}{3\pi(\beta+1)}\right]^{1/2},$$

$$Sh_\infty = \frac{3}{8}\frac{(3\pi)^{2/3}}{\Gamma(1/3)} Pe^{1/3} = 0.624\, Pe^{1/3}.$$

Here and hereafter the index Pe at the respective asymptotic expressions of the mean Sherwood number will be omitted.

The consequence of the above results being non-uniform with respect to the parameter β are the properties

$$Sh_\infty \neq \lim_{\beta \to \infty} Sh_\beta = 0.$$

It has been shown [6, 7] that the range of validity of the formulae obtained in the diffusional boundary-layer approximation for the droplets with $Sh = Sh_\beta$, relations (7), is bounded above in the parameter β by the inequality $\beta < 0(Pe^{1/3})$. This inequality follows, for example, from the two-term expansion of the mean

Sherwood number in the Peclet number in the limit for $Pe \to \infty$ (β = const.) [7]

$$Sh = \left[\frac{2Pe}{3\pi(\beta+1)}\right]^{1/2} + 0.413(\tfrac{3}{4}\beta+1). \quad (8)$$

The error of equation (8) has the order of $Pe^{-1/2}$. It is the allowance for the fact that the first term in expansion (8), which coincides with Sh_β in equation (7), must be much in excess of the second term, which leads to the sought-after inequality given above.

The said problem of mass transfer of a droplet has been studied [8] by the approximate integral method over the entire range of droplet viscosities, $0 \leqslant \beta \leqslant \infty$, with additional physical concepts being used which are not directly associated with the convective diffusion equation.

Now, it will be shown how the above procedure can improve the results of work [8], where the following approximate cubic equation was obtained for the mean Sherwood number

$$Sh^3 - \frac{Pe}{16 \ln 2(\beta+1)}(2Sh+3\beta+1) = 0. \quad (9)$$

For the mean Sherwood number defined by equation (9), there are two main asymptotic expressions

$$\beta = 0(1),\ Sh = Sh_\beta^*; \quad \beta \to \infty,\ Sh \to Sh_\infty^* \quad (10)$$

$$Sh_\beta^* = \left[\frac{Pe}{8 \ln 2(\beta+1)}\right]^{1/2}, \quad Sh_\infty^* = \left(\frac{3Pe}{16 \ln 2}\right)^{1/3},$$

that correspond to a moderately viscous droplet, $\beta = 0(1)$, and a solid sphere, $\beta = \infty$; in deriving relations (10) it is allowed for that the Peclet number is large, $Pe \gg 1$.

It is seen that approximate expressions (10) differ from asymptotically accurate relations (7) only by numerical factors.

Now, from equations (10) express the parameters β and Pe in terms of Sh_β^* and Sh_∞^* and substitute these into equation (9). This will give

$$Sh^3 - Sh_\beta^{*2}(Sh-1) - Sh_\infty^{*3} = 0. \quad (11)$$

The exact asymptotic expressions, corresponding to expressions (10), for the solution of the initial problem are given above and have the form of relations (7).

Taking into account the fact that at large Peclet numbers $Sh \gg 1$ and $Sh - 1 \approx Sh$ and replacing Sh_β^* and Sh_∞^* in equation (11) by Sh_β and Sh_∞ from equation (7) will yield the following approximate equation to determine the mean Sherwood number

$$Sh^3 - Sh_\beta^2 Sh - Sh_\infty^3 = 0, \quad (12)$$

which correctly reflects the quantitative behaviour of Sh and leads to an exact asymptotic result in the limiting cases at $\beta = 0(1)$ and $\beta = \infty$ ($Pe \gg 1$).

The mean Sherwood number, defined by equation (12), can be presented in the following form more convenient for the analysis

$$Sh = Sh_\beta x(Q),$$

$$Q = Sh_\infty/Sh_\beta = \text{const.} \times Pe^{-1/6}(\beta+1)^{1/2}, \quad (13)$$

where Q is the similarity parameter and $x = x(Q)$ is the root of the equation

$$x^3 - x - Q^3 = 0.$$

It will be shown further that equation (12) can also be successfully used to approximately determine the mean Sherwood number per arbitrarily viscous droplet in a linear shear flow.

Figure 1(a) presents a correlation between the root of approximate equation (12) (solid lines) and the available asymptotic, equations (7), (8), and numerical [9] solutions (dashed–dotted and dashed lines, respectively) for this particular problem of mass transfer of a droplet at $Pe = 500$ (lower curves) and at $Pe = 5000$ (upper curves). It is seen that approximate equation (12) gives an adequate description of the mean Sherwood number behaviour at large Peclet numbers over the entire range of the droplet and surrounding liquid viscosity ratio. Some difference between the root of equation (12) and numerical results of work [9] at small values of the parameter β results from the unsuitable choice of the finite-difference grid step in the latter reference. This is evident, in particular, from a qualitative comparison of the asymptotically accurate

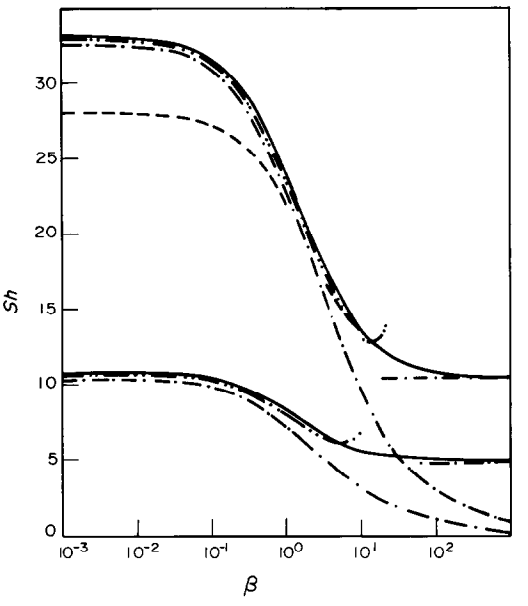

FIG. 1(a). The mean Sherwood number Sh vs the droplet to surrounding liquid viscosity ratio β at large Peclet numbers: ——, the results obtained by solving cubic equation (12); – – –, numerical calculations by the finite-difference method [12] {at $Pe = 500$ the results obtained from equation (12) and the data of ref. [12] practically coincide and are represented here by one solid line}; – · – · –, asymptotic expressions (7) obtained in the diffusional boundary-layer approximation [5]; – · · – · · –, two-term expansion of the mean Sherwood number in Peclet number, equation (8) [7].

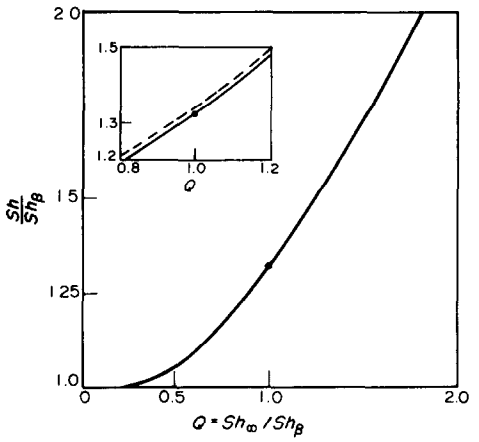

FIG. 1(b). The normalized mean Sherwood number Sh/Sh_β vs the similarity parameter $Q = Sh_\infty/Sh_\beta$: ——, the results obtained by solving cubic equation (12); – – –, the results obtained by calculating the integral of equation (71).

formula (8) with the results of ref. [9]. Namely, it follows from equation (8) that at $\beta \approx 0$ and for $Pe \to \infty$ the true curve should be located somewhat above the solution $Sh = Sh_\beta$, equation (7), and not below it, as is the case in ref. [9]. This will be discussed in more detail in Section 8 where equation (12) will be derived quite independently by another method. Figure 1(b) shows the dependence of the Sherwood number on the similarity parameter $Q = Sh_\infty/Sh_\beta$.

Consider one other specific example when the proposed procedure allows a substantial improvement of the approximate formula. In ref. [10], a steady-state convective diffusion with simultaneous chemical reaction of an arbitrary order n was investigated. For the mean Sherwood number, corresponding to a translational Stokes flow past a spherical droplet [the Hadamard–Rybczinskii flow, stream function (65)], the following equation was obtained

$$Sh = Sh(k_v, Pe) = \left(\frac{1}{8\ln 2}\frac{Pe}{\beta+1} + \frac{2k_v}{n+1}\right)^{1/2}, \quad (14)$$

where k_v is the dimensionless chemical reaction rate constant. Equation (14) was derived by the integral method and it was assumed that $Pe \gg 1$. Now, the parameters to be used for the improvement of approximate formula (14) are selected to be the quantities $k = k_v$, $p = Pe$.

When $k_v \to \infty$ (Pe = const.), approximate expression (14) correctly describes the asymptotic behaviour of the mean Sherwood number, while in the other limiting case

$$k_v \to 0, \quad Sh \to Sh_0 = Pe^{1/2}[8\ln 2(\beta+1)]^{-1/2}. \quad (15)$$

Here and hereafter, the asterisk at the respective asymptotic expressions of the approximate solution will be omitted; $Sh_0 = Sh(0, Pe)$. Now, with the aid of equation (15), express the Peclet number in terms of Sh_0 and substitute it into equation (14). This yields the following expression

$$Sh = \left(Sh_0^2 + \frac{2k_v}{n+1}\right)^{1/2}. \quad (16)$$

When this formula is used for Sh_0, an exact value of the mean Sherwood number should be selected which would correspond to a pure diffusional regime not involving a chemical reaction ($k_v = 0$). It should be noted that the validity range of the improved formula (16) turns to be much wider that that of the initial approximate formula (14); it may already be used to approximately determine the mean Sherwood number for both an arbitrarily shaped droplet and solid particle in an arbitrary flow in the case of large Peclet numbers. In particular, for a translational flow past a solid sphere it is necessary to assume in equation (16) that $Sh_0 = Sh_\infty$, where Sh_∞ has been defined in equation (7).

Note, that in this case the parameter k_v has not been eliminated through the use of the asymptotic expression of approximate formula (14) when $k_v \to \infty$ because it coincided with the respective exact asymptotic expression.

It should be noted that, when required, the asymptotic correction can be repeated several times. In particular, it will be shown further in Section 6 in which way formula (16) can thus be improved that it can be used also at small and moderate Peclet numbers.

In the case of a translational Stokes flow past a spherical droplet with a first-order chemical reaction, $n = 1$, the comparison between the approximate formula (16) and the results of numerical integration [11] of the earlier obtained (in the diffusion boundary-layer approximation) expression for the local diffusion flux [12] is presented in Fig. 2 {the solid line corresponds to

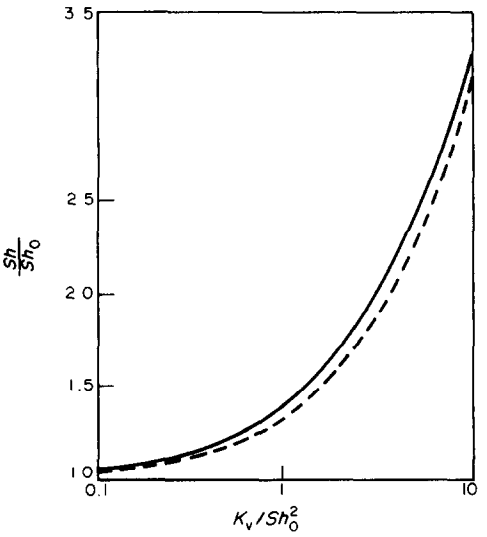

FIG. 2. The mean Sherwood number vs the reaction rate constant and Peclet number for the first-order volumetric chemical reaction: ——, the results of calculation by formula (16) at $n = 1$; – – –, the results of calculations obtained in the diffusional boundary-layer approximation for a spherical droplet in a translational Stokes flow [11, 12].

formula (16) and the dashed line to the results of refs. [11, 12]}. It is seen that in this case the maximum error of equation (16) over the entire range of the dimensionless volumetrical reaction rate constant k_v is about 7%.

A similar comparison with the results of numerical analysis carried out in ref. [13] for a fractional-order reaction, $n = 1/2$, in the case of a translational Stokes flow past a spherical droplet of small ($\beta = 0.01$) and high ($\beta = 10$) viscosity at large Peclet numbers shows that the maximum error of equation (16) in this case is about 5%. It should be noted, however, that in the work mentioned the convective terms in the diffusion equation were included completely (without stream function linearization), and the Laplace operator, which corresponds to a molecular diffusion, was replaced only by its spherical part (i.e. the Δ operator was replaced by $\partial^2/\partial r^2 + 2r^{-1}\partial/\partial r$), with a wrong boundary condition, which is 'swept' from infinity, being specified on the flow axis at $\theta = 0$. It is not difficult to show that in the problem stated as such, the concentration on the flow axis is not identically constant, which is governed by its valuc at infinity (i.e. $c \neq 0$ at $\theta = 0$), and need be specified as

$$\theta = 0, \quad c = c_0(r), \tag{17}$$

where the function $c_0(r)$ is the solution of the following ordinary differential equation

$$k_v c_0^n + Pe\, v_r(r, 0)\frac{dc_0}{dr} = \frac{d^2c_0}{dr^2} + \frac{2}{r}\frac{dc_0}{dr}; \tag{18}$$

$$r = 1,\ c_0 = 1; \quad r \to \infty,\ c_0 \to 0,$$

which is the corollary of the partial differential equation due to the symmetry condition on the flow axis ($\partial c/\partial\theta = 0$ at $\theta = 0$) [13].

The above inaccuracy could naturally introduce substantial errors in the results of numerical analysis [13]. It should be noted that in the absence of a volumetric chemical reaction, a correct boundary condition (17) corresponding to solution (18) at $k_v = 0$ was adopted in work [9].

3. THE METHOD OF MODEL EQUATIONS AND ANALOGIES

In the majority of problems encountered, the main practical interest attaches usually not to a complete solution of the problem but only to some of its mean integral characteristics S (e.g. in the case of mass and heat transfer between particles and a flow, such main characteristics of the process are the mean Sherwood and Nusselt numbers, while direct determination of concentration and temperature fields often turns to be a secondary information of little importance). Below, a sufficiently general method is suggested for the construction of approximate formulae for the quantity S, where the method is based on a secondary, much more simple, problem than the initial one.

Let there be a complex non-linear boundary-value problem described by a partial-differential equation and dependent (for simplicity) on two dimensionless parameters, k and p (e.g. the Damköler and Peclet numbers). Then it is assumed that it is impossible to obtain a complete exact solution of the initial problem, and it is necessary to construct approximate formulae for the integral characteristic $S = S(k, p), 0 \leqslant k, p < \infty$.

First, the most simple version of the method will be described.

It is assumed that at $k = 0$ (or $k = \infty$) the initial problem is simplified (e.g. becomes a linear one) and can be solved, i.e. the quantity

$$S_0 = S_0(p) \equiv S(0, p) \tag{19}$$

is known.

It is assumed that at $k = 0$ the problem does not degenerate, retains its type and its solution is non-trivial (i.e. it not identically constant).

In order to construct an approximate solution of the problem, consider, instead of the initial partial differential equation, its one-dimensional analogue (the model equation) which can already be described by an ordinary differential equation. Construct the solution of the model equation and obtain for this the analogue, $s = s(k, p)$, of the integral characteristic S (local diffusional flux). The assumption that $k = 0$ yields the expression $s_0 = s_0(p) \equiv s(0, p)$. Then, the elimination of the parameter p from $s = s(k, p)$ and $s_0 = s_0(p)$ results in

$$G(s, s_0, k) = 0. \tag{20}$$

Now, formulate the *analogy principle*, i.e. assume that the functional relation of the corresponding integral characteristics of the initial problem, S and S_0, is analogous to relation (20) for the local characteristics of the model problem, s and s_0, i.e. the formula

$$G(S, S_0, k) = 0 \tag{21}$$

is valid, where $S_0 = S_0(p)$ is determined by equation (19). The required quantity $S = S(k, p)$ is found by solving the algebraic equation (21). It is evident from approximate equation (21) that at $k = 0$ its solution will coincide with exact solution (19).

Although the method described above has been formulated heuristically, it can be easily put in a more general and formal form. Namely, its basic ideas can be reformulated as follows. First, in order to construct an approximate solution of the problem, instead of the initial partial differential equation, consider its one-dimensional analogue (model equation) which is already described by a much more simple ordinary differential equation. Then, having constructed the solution of the model equation, use it to determine the dependence of the local analogue of the mean integral characteristic s (local diffusional flux) on the parameters k and p. It is the resulting expression $s = s(k, p)$ which is taken to be the very initial approximate formula (1) (at $s \equiv S$) which should be improved by the method of asymptotic correction described in Section 2 (i.e. here the 'identification' of s and S is made), on the assumption that two exact asymptotic expressions of

the full initial problem of type (4) known (in the earlier simplest version of the method only one asymptotic expression has been used for correction).

Now, the possibilities of the method of model equations and analogies will be illustrated as applied to the problems of convective mass and heat transfer between particles and a flow in the presence of surface or volumetric chemical reactions.

4. MASS TRANSFER OF A MOVING PARTICLE IN THE PRESENCE (ON ITS SURFACE) OF A CHEMICAL REACTION WITH ARBITRARY KINETICS

Consider convective diffusion to a reacting solid or liquid spherical particle in a laminar flow of viscous incompressible liquid in the presence on its surface of an isothermal chemical reaction the rate of which is arbitrarily dependent on concentration.

The distribution of the active component concentration in the flow is governed by the following dimensionless equation and boundary conditions

$$\Delta c = Pe(\mathbf{v}\cdot\nabla)c; \quad r\to\infty,\ c\to 0 \tag{22}$$

$$r=1, \quad -\frac{\partial c}{\partial r} = k_s f_s(c) \tag{23}$$

$$c=\frac{C_\infty - C_*}{C_\infty}, \quad Pe=\frac{aU}{D}, \quad k_s=\frac{aK_sF_s(C_\infty)}{DC_\infty},$$

$$f_s(c)=\frac{F_s(C_*)}{F_s(C_\infty)}.$$

Here, C_* and C_∞ are the concentration in the flow and at infinity, respectively; a is the particle radius; U the characteristic flow velocity; D the diffusion coefficient; K_s the surface reaction rate constant; K_sF_s the surface reaction rate; $\mathbf{v}$ the dimensionless liquid velocity vector (it is regarded to be known from the solution of the corresponding hydrodynamic problem; see below); Δ the Laplace operator; r, θ, ϕ the spherical coordinate system with the origin located at the particle center; the condition $F_s(0)=0[f_s(1)=0]$ is assumed to be fulfilled.

The parameters k and p in this problem are

$$k = 1/k_s, \quad p = Pe, \tag{24}$$

and the unknown integral quantity S is the mean Sherwood number over the sphere surface

$$Sh = -\frac{1}{2}\int_0^\pi \left(\frac{\partial c}{\partial r}\right)_{r=1} \sin\theta\, d\theta \tag{25}$$

[of course, it was possible to take the inverse quantity, $k = k_s$, for the parameter k, but the choice of k by formula (24) is more convenient to preserve the uniformity of notation].

The limiting transition in the problem (22), (23) when $k_s \to 0$ leads to the trivial solution $c = 0$, which is not suitable for the use of the method. The other limiting transition in equation (23) for $k_s \to \infty$ (or $k \to 0$) shows, by virtue of the properties of the function f_s ($f_s(1)=0$), that the auxiliary function $Sh_0 = Sh_0(Pe)$, equation (19), should be determined by solving the problem (22) subject to the simplest boundary condition on the sphere surface

$$r = 1, \quad c = 1. \tag{26}$$

Note that the problem (22), (26) is linear and is much simpler than the initial non-linear problem (22), (23). At present there are sufficient number of the solutions to problem (22), (26), which have been obtained by numerical, asymptotic or approximate methods for different flows past a droplet or a particle. Moreover, as Sh_0 one may also use empirical formulae obtained as a result of the treatment of experimental data. Further it is assumed that the function $Sh_0 = Sh_0(Pe)$ is known.

The simplest one-dimensional analogue of equation (22) is an ordinary differential equation with constant coefficients

$$c''_{\xi\xi} + Pe\,c'_\xi = 0; \quad \xi\to\infty,\ c\to 0 \tag{27}$$

$$\xi = 0, \quad -c'_\xi = k_s f_s(c), \tag{28}$$

which has been obtained by approximating the coefficients of the initial partial differential equation with the variable coefficients (22) by their values at infinity in the case of translational flow ($\mathbf{v}\to -\mathbf{i}$, $\mathbf{i}$ is the unit vector of axis x) with a subsequent 'replacement' of the curvilinear surface of the sphere $r = 1$ by the tangential plane $x = 1$ ($x = r\cos\theta$) at the incidence point and by further replacement $\xi = x - 1$.

The general solution of equation (27) is

$$c = A\exp(-Pe\,\xi) \quad (A = \text{const.}). \tag{29}$$

The substitution of this expression into boundary condition (28) leads to the following equation to determine the coefficient A

$$A\,Pe = k_s f_s(A). \tag{30}$$

The analogue of the mean Sherwood number (25) in model problem (27), (28) is the local diffusional flux $j = j(k_s, Pe) = -(c'_\xi)_{\xi=0}$. Taking into account expressions (29), (30) for j, one obtains

$$j = k_s f_s(j/Pe). \tag{31}$$

Now, letting k_s go to infinity (i.e. assuming $k = 1/k_s = 0$) and taking into account the properties of the function f_s ($f_s(1)=0$) result in

$$j_0 = Pe. \tag{32}$$

The elimination of Pe from relations (31) and (32) yields the expression

$$j = k_s f_s(j/j_0). \tag{33}$$

From this, by virtue of the analogy principle, the following (algebraic) equation is obtained for the determination of the mean Sherwood number

$$Sh = k_s f_s(Sh/Sh_0), \quad Sh_0 = Sh_0(Pe). \tag{34}$$

A direct analysis of the boundary-value problem (22),

(23) shows that an approximate equation for the mean Sherwood number (34) gives a correct asymptotic result in all the limiting cases at large and small parameters k_s and Pe.

Evaluate now the accuracy of the approximate equation (34) by comparing its solutions with the well-known exact, approximate and asymptotic results.

4.1. *Small Peclet numbers*

It has been shown [14–16] that approximate equation (34) is an asymptotically exact one and makes it possible, in the case of a translational and shear flow past a sphere, to obtain at least three terms of the corresponding expansion in small Peclet number for any surface chemical reaction kinetics within the entire range of the dimensionless reaction rate constant k_s.

4.2. *Intermediate Peclet numbers*

The applicability of approximate equation (34) at intermediate Peclet numbers in the case of translational viscous incompressible liquid flow past a sphere was verified by comparing with the well-known results of the numerical solution to the corresponding problem for the first-order heterogeneous reaction ($F_s(C_*) = C_*$) at $Pe = 10, 20, 50$ [17]. Here too, the flow field was determined by the finite-difference methods, with the Peclet numbers corresponding to the following Reynolds numbers: $Re = 10, 20, 0.5$. It follows from the table given in ref. [17] that the use of approximate equation (34) leads to very good results (error does not exceed 1.5%).

4.3. *Large Peclet numbers*

At large Peclet numbers (in the diffusional boundary-layer approximation) in the case of the reactions of the order of $n = 1/2, 1, 2$ ($F_s = C_*^n$), the validity of interpolation equation (34) was checked over the entire range of the parameter $k_s = aK_sD^{-1}C_\infty^{n-1}$ by comparing its root Sh with the exact results obtained for the mean Sherwood number by numerical integration of the corresponding integral equations for the local diffusional flux in the case of translational Stokes flow past a sphere [18], circular cylinder [19], droplet and bubble [20]. In all of these cases, the characteristic length scale was chosen to be the radius of the sphere, cylinder and droplet, a, and the characteristic velocity U—the velocity of flow at infinity, U_∞. The results of comparison between the exact and approximate values of the Sherwood number show that the maximum deviation of the root of equation (34) from the exact value of the mean Sherwood number [18–20] does not exceed 12%.

The results of comparison of the exact [18–20] and approximate [calculated by equation (34)] values of the mean Sherwood number on an example of the second-order ($n = 2$) reaction are presented in Fig. 3 [for $n = 1/2$ and $n = 1$ the accuracy of equation (34) is higher than for $n = 2$]. It is seen that in this case the maximum uncertainty is observed at $k_s/Sh_0 \approx 0.5$–5.0 and does not exceed 6% for a solid sphere, 8% for a circular cylinder and 12% for a spherically shaped droplet (bubble).

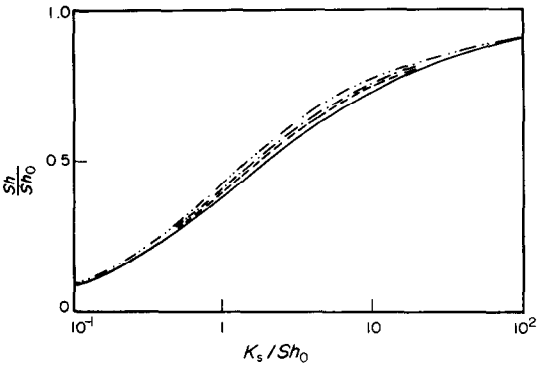

FIG. 3. The mean Sherwood number vs the rate constant and Peclet number for the second-order surface chemical reaction ($F_s = C_*^2$): —, the results of calculation by formula (34); – – –, the results obtained in the diffusional boundary-layer approximation by numerical integration of the corresponding non-linear integral equation for a local Stokes flow past a solid sphere [18]; – · – · –, the results of calculations obtained in the diffusional boundary-layer approximation for a translational Stokes flow past a circular cylinder [19]; – · · – · · –, the results of calculations obtained for a Stokes flow past a droplet [20].

It follows from the results of work [16] that with the characteristic length scale, a, selected properly, equation (34) can also be successfully used to calculate the mass transfer of random shaped particles.

The comparisons mentioned show a good accuracy and wide possibilities of approximate equation (34).

5. ONE EXACT EXAMPLE OF ANALOGY CONSERVATION

Besides problem (27), (28), the model problem, which would correspond to problem (22), (23), can also be constructed on the basis of other, more specific considerations. Namely, with large Peclet numbers, the model problem is taken to be the problem of concentration distribution in the vicinity of the forward stagnation point (the point of incidence $\theta = 0$) of a spherical droplet in a translational Stokes flow [the Hadamard–Rybczinskii flow, formula (65)]

$$c''_{\xi\xi} + Pe\, v(\xi) c'_\xi = 0; \quad \xi \to \infty,\ c \to 0 \tag{35}$$

$$\xi = 0, \quad -c'_\xi = k_s f_s(c) \quad (\xi = x - 1). \tag{36}$$

The derivation of equation (35) and the relation $v = v(\xi)$ will be given further in Section 7. In particular, for a solid sphere $v(\xi) = \frac{3}{2}\xi^2$; further, for generality, the form of the function $v = v(\xi)$ will not be specified.

The general solution of equation (35) is

$$c = A \int_\xi^\infty \exp\left(-Pe \int_0^\zeta v(\tau)\, d\tau\right) d\zeta \quad (A = \text{const.}). \tag{37}$$

The substitution of equation (37) into boundary condition (36) yields the following equation for the

parameter A

$$A = k_s f_s\left(\frac{A}{\Omega}\right),$$

$$\Omega = \left\{\int_0^\infty \exp\left(-Pe\int_0^\zeta v(\tau)\,d\tau\right)d\zeta\right\}^{-1}. \quad (38)$$

Here, the analogue of the mean Sherwood number is the local diffusional flux at the forward stagnation point of the particle, $j = j(k_s, Pe) = -(c'_\xi)_{\xi=0}$.

Equation (37) gives relationship $A = j$ which, on being substituted into equation (38), yields the equation for the local flux

$$j = k_s f_s(j\Omega^{-1}). \quad (39)$$

Passing to the limit for $k_s \to \infty$ in equation (39) and taking into account the property $f_s(1) = 0$ result in the equality

$$j_0 = \Omega. \quad (40)$$

The substitution of this expression into equation (39) gives the functional relationship between j and j_0, which coincides exactly with equation (33); by virtue of the analogy principle, this leads, in turn, to approximate equation (34). This example shows that in spite of the fact that model problems (27), (28) and (35), (36) differed substantially and were based on different arguments, the sought-after functional relationship for them was the same, equation (33).

On practical grounds, it is, of course, more convenient that the model equation is taken to be the most simple equation with constant coefficients, i.e. equation (27).

It should be noted that the simplest problem (27), (28) may be regarded as a model for a more complex problem (35), (36). For this, it is sufficient in equations with variable coefficients (35) to formally assume that $v(\xi) \equiv 1$. In this case, an approximate relationship between j and j_0 (33), obtained by solving the model problem (27), (28), will coincide with the exact relationship, which is found by solving the initial problem (35), (36) (i.e. in this case the analogy principle holds).

6. CONVECTIVE MASS TRANSFER TO A DROPLET OR SOLID PARTICLE IN THE PRESENCE OF AN ARBITRARY VOLUMETRIC CHEMICAL REACTION IN THE LIQUID

In the case of a volumetric chemical reaction occurring with an arbitrary kinetics in the liquid, the problem of mass transfer between a particle and a flow is formulated as

$$\Delta C = Pe(\mathbf{v}\cdot\nabla)C + k_v f_v(C) \quad (f_v(0) = 0) \quad (41)$$

$$r = 1,\ C = 1;\quad r \to \infty,\ C \to 0 \quad (42)$$

$$C = \frac{C_*}{C_s},\quad k_v = \frac{a^2 K_v F_v(C_s)}{DC_s},\quad f_v(C) = \frac{F_v(C_*)}{F_v(C_s)}.$$

Here C_s is the concentration on the particle surface, K_v the volumetric chemical reaction rate constant, $K_v F_v$ the chemical reaction rate; the remaining parameters are described in Section 4.

In this problem, the parameters k and p are

$$k = k_v,\quad p = Pe \quad (43)$$

and the analogue of S is the mean Sherwood number, which is defined by equation (25) on replacing c by C. The auxiliary function $Sh_0 = Sh_0(Pe)$, equation (19), is found by solving the linear problem (41), (42) at $k_v = 0$ [or problem (22), (26)]. Further, just as in Section 4, it is assumed that the function Sh_0 is known.

The simplest one-dimensional analogue of problem (41), (42) is

$$C''_{\xi\xi} + PeC'_\xi = k_v f_v(C) \quad (44)$$

$$\xi = 0,\ C = 1;\quad \xi \to \infty,\ C \to 0. \quad (45)$$

In contrast to equation (27), it is impossible to obtain the general solution of equation (44) for the arbitrary form of the function $f_v(C)$.

In order to construct the required functional relationship between the local fluxes, equation (20), the approximate method of integration of equation (44) will be employed. Introduce the new variable $u = C'_\xi$. Then equation (44) can be written as $du^2 + Pe\,u\,dC = 2k_v f_v(C)\,dC$. Integrate it over C from zero to C taking into account the fact that when $\xi \to \infty$, $u = C'_\xi \to 0$. This results in

$$u^2 = -2Pe\int_0^C u\,dC + 2k_v\int_0^C f_v(C)\,dC. \quad (46)$$

The solution of the integral equation (46) will be sought by the method of iterations according to the formula

$$u_n = -\left(-2Pe\int_0^C u_{n-1}\,dC + 2k_v\int_0^C f_v(C)\,dC\right)^{1/2};$$

$$n = 1, 2, \ldots, \quad (47)$$

where any negative function can be chosen for the initial profile $u_0 = u_0(C)$.

The restriction in equation (47) to one iteration and the assumption that $\xi = 0$, with $C(0) = 1$ and $j = -u(1)$ taken into account, yield the following expression for the local flux

$$j = \left(j_0^2 + 2k_v\int_0^1 f_v(C)\,dC\right)^{1/2},$$

$$j_0^2 = -2Pe\int_0^1 u_0(C)\,dC. \quad (48)$$

From this, by virtue of the analogy principle (20), (21) and equation (43), the mean Sherwood number is defined as

$$Sh = (Sh_0^2 + 2k_v\langle f_v\rangle)^{1/2},\quad \langle f_v\rangle \equiv \int_0^1 f_v(C)\,dC. \quad (49)$$

The approximate expression for the mean Sherwood

number (49) in the limiting cases of large and small values of Peclet number and parameter k_v will now be investigated.

As already pointed out in Section 3, at $k_v = 0$ formula (49) (by construction) yields the exact result $Sh_0(Pe)$.

At large Peclet numbers, the second term on the RHS of equation (41) can be neglected, therefore when $Pe \to \infty$ (k_v = const.) the main term of the asymptotic expression for Sh coincides with Sh_0; this very limiting relationship is also obtained from formula (49), since when $Pe \to \infty$, $Sh_0 \to \infty$. (It should be noted that the cases when the particle is surrounded by the flow region with completely closed stream lines, similar to those analyzed in ref. [21], are not included in this consideration.)

When $k_v \to \infty$ (Pe = const.), the first term on the RHS of equation (41) can be neglected. In this case, the introduction of a new 'stretched' variable, $z = k_v^{1/2}(r-1)$, into equation (41) with the subsequent separation of the main terms of expansion in large parameter k_v leads to equation (44) at $Pe = 0$ and $\xi = r-1$. The integration of the resulting ordinary differential equation provides a quadratic equation (46) for u at $Pe = 0$. In this limiting case the local flux does not depend on the angular coordinate θ and coincides with the mean Sherwood number $j = Sh$. In light of what has been said, the comparison of the main term of expansion (49) for $k_v \to \infty$ and equation (46) at $Pe = 0$ shows that approximate formula (49) gives a correct asymptotic result when $k_v \to \infty$.

In the particular case of the power-law dependence of the volumetric chemical reaction rate on concentration, $f_v(C) = C^n$, formula (49) passes over into equation (16), which has been obtained earlier by quite a different method. From the comparison, carried out earlier in Section 2, of formula (49) with the results of numerical calculations for some particular cases and also from the above-mentioned limiting properties it follows that approximate expression (49) can be used with success to calculate the mean Sherwood number at large Peclet numbers over the entire range of the dimensionless volumetric chemical reaction rate constant k_v.

Formula (49) can also be employed to calculate the convective mass transfer of arbitrarily shaped droplets and particles. Then the mean Sherwood number should be determined from

$$Sh = \frac{I}{\Sigma}, \quad I = -\int_\Gamma \left(\frac{\partial C}{\partial n}\right) d\Gamma, \quad \Sigma = \int_\Gamma I \, d\Gamma, \tag{50}$$

where I and Σ is the dimensionless integral diffusional flux and particle surface area, respectively (Γ is the particle surface). With the mean Sherwood number defined this way, approximate expression (49) provides a correct asymptotic result in the limiting cases $k_v \to 0$ (Pe = const.); $k_v \to \infty$ (Pe = const.); $Pe \to \infty$ (k_v = const.).

Formula (49) can also be used for the calculation of mass transfer between droplets and particles at large Peclet numbers and the flows of complex geometry, when the distribution of liquid velocities at infinity is not uniform. In particular, in the case of an axisymmetric linear shear flow past a solid spherical particle, it is necessary to assume in equation (49) that $Sh_0 = Sh_\infty$, where Sh_∞ has been defined by equation (74).

As has frequently been remarked, the approximate formula (49) gives good results only at large Peclet numbers. However, it should be noted that expression (49) can easily be improved by resorting again to the method of asymptotic correction. Namely, assume in equation (49) that $Pe = 0$. This, in the case of a droplet or spherically shaped particle, will give the following expression

$$\hat{Sh} = (1 + 2k_v \langle f_v \rangle)^{1/2}. \tag{51}$$

Now the elimination of the parameter k_v from equations (49) and (51) will lead to the following expression for the mean Sherwood number

$$Sh = (Sh_0^2 + \hat{Sh}^2 - 1)^{1/2}. \tag{52}$$

When further using this formula, it will be assumed that here

$$\hat{Sh} \equiv \hat{Sh}(k_v) = Sh(k_v, Pe = 0) \tag{53}$$

is already determined by the exact solution of auxiliary problem (41), (42) at $Pe = 0$.

Formula (52) gives correct asymptotic result already in all the limiting cases of large and small values of the parameters k_v and Pe (at $Pe = 0$ including) for any function $f_v = f_v(C)$. This can easily be shown by using the relations $Sh_0(0) = 1$, $\hat{Sh}(0) = 1$ and $Pe \to \infty$, $Sh_0 \to \infty$; $k_v \to \infty$, $\hat{Sh} \to \infty$. The approximate formula (52) should be employed in those cases when the relation $\hat{Sh} = \hat{Sh}(k_v)$ is known. Thus, in the linear case $f_v(C) = C$, the exact solution of problem (41), (42) at $Pe = 0$ is

$$C = r^{-1} \exp\{-k_v^{1/2}(r-1)\}, \tag{54}$$

which corresponds to the following expression for an auxiliary mean Sherwood number

$$\hat{Sh} = 1 + \sqrt{k_v} \tag{55}$$

from equation (52).

It should be noted that the knowledge of exact approximations for the asymptotes to the solution of problem (41), (42) for k_v, $Pe = 0$ and k_v, $Pe \to \infty$ is insufficient for the construction of approximate formulae which would give a good description of the process at arbitrary values of the parameters k_v and Pe. For example, instead of equation (52) it is possible to suggest a much simpler formula $Sh = Sh_0 + \hat{Sh} - 1$, which also provides a correct asymptotic result in all limiting cases. However, the comparison of this formula for the linear case, equation (55), with the results of ref. [10] proves its absolute inadequacy, since the error in this case amounts to more than 50% [recall that the maximum error of equations (52), (55) is about 7%].

In addition to correct asymptotic expressions, the proposed method of model equations and analogies seems to be capable of 'grasping' also the internal structure of the solution, which ultimately enables, in the problems of convective mass and heat transfer of particles with surface and volumetric reactions, a rather accurate result to be obtained over the entire range of characteristic parameters k and Pe (with an arbitrary chemical reaction kinetics).

7. UNSTEADY-STATE CONVECTIVE MASS AND HEAT TRANSFER TO A DROPLET OR SOLID PARTICLE

Now consider the unsteady-state convective mass transfer to a solid or liquid spherical particle reacting in the diffusive regime. In terms of dimensionless variables, the corresponding boundary-value problem is formulated as

$$\frac{\partial c}{\partial t} + Pe(\mathbf{v}\cdot\nabla)c = \Delta c \tag{56}$$

$$t = 0,\ c = 0;\quad r = 1,\ c = 1;\quad r \to \infty,\ c \to 0 \tag{57}$$

$$C = \frac{C_\infty - C_*}{C_\infty},\quad t = \frac{Dt_*}{a^2},$$

where t_* is the time; it is further assumed that the distribution of liquid velocities $\mathbf{v}$ in equation (56) is known and corresponds to some steady-state regime of flow outside the particle.

In order to analyze problem (57), the method of model equations and analogies will be applied. As a one-dimensional analogue of problem (57), use the following model equations and boundary conditions

$$\frac{\partial c}{\partial t} - Pe\,\xi\frac{\partial c}{\partial \xi} = \frac{\partial^2 c}{\partial \xi^2} \quad (\xi = x - 1) \tag{58}$$

$$t = 0,\ c = 0;\quad \xi = 0,\ c = 1;\quad \xi \to \infty,\ c \to 0,$$

which describe the distribution of concentration in the vicinity of the forward stagnation point of a spherical bubble [the stream function of such a flow is given by equation (65) at $\beta = 0$] at large Peclet numbers.

The solution of problem (58) is sought in the form

$$c = c(u),\quad u = \frac{\xi}{2\sqrt{\delta}},\quad \delta = \delta(t).$$

The substitution of these expressions into equation (58) shows that the unknown functions $c = c(u)$ and $\delta = \delta(t)$ satisfy the following equations and boundary conditions

$$c''_{uu} + 2uc'_u = 0;\quad c(0) = 1,\ c(\infty) = 0 \tag{59}$$

$$\delta'_t + 2Pe\,\delta = 1;\quad \delta(0) = 0.$$

The solution of problems (59) determines the unsteady-state concentration distribution in the vicinity of the forward stagnation point of the bubble as

$$c = \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\delta}}\right),\quad \delta = \frac{1}{2Pe}\{1 - \exp(-2Pet)\}.$$

This formula gives the time dependence of the local diffusional flux

$$j = (2Pe/\pi)^{1/2}\{1 - \exp(-2Pet)\}^{-1/2}. \tag{60}$$

Now, letting in this equation t go to infinity, it is possible to obtain that

$$j_0 = \lim_{t\to\infty} j = (2Pe/\pi)^{1/2}.$$

Further, expressing the Peclet number Pe in terms of j_0 and substituting it into equation (60), results in

$$j/j_0 = \{1 - \exp(-\pi j_0^2 t)\}^{-1/2}.$$

From this, by virtue of the analogy principle, the following formula can be obtained for the mean Sherwood number

$$\frac{Sh}{Sh_0} = \{1 - \exp(-\pi Sh_0^2 t)\}^{-1/2}\ (Sh_0 = \lim_{t\to\infty} Sh). \tag{61}$$

The approximate dependence of the mean Sherwood number (61) on time is of rather a general character and may presumably be used for the calculation of the unsteady-state mass and heat transfer of arbitrarily shaped solid particles, droplets and bubbles [in a general case the mean Sherwood number is determined from equation (50)] in flows of different geometry at large Peclet numbers. Figure 4 presents the comparison between approximate expression (61) (solid line) and the results of works [22–26] obtained in the diffusional

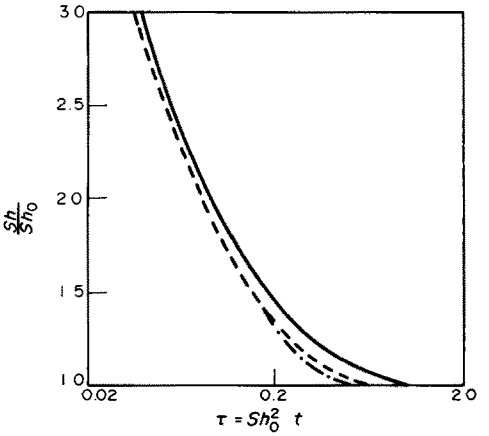

FIG. 4. The normalized mean Sherwood number vs time: ——, the results of calculation by formula (61); – – –, the results of calculation by formula (96) corresponding to unsteady-state mass transfer to a spherical droplet in an axisymmetric shear Stokes flow [25] and also the results of calculation for a translational Stokes flow past a droplet [22–24] and a translational non-viscous liquid flow past a sphere [23, 24] (corresponding results differ very little and are represented here by one dashed line); – · – · –, the results of calculations for the case of a translational Stokes flow past a solid sphere [26].

boundary-layer approximation for a moderately viscous spherical droplet streamlined by a translational [22–24] and axisymmetric linear shear [25] flow (dashed line), and also for a translational Stokes flow past a solid spherical particle [26] (dashed–dotted line). The auxiliary mean Sherwood number Sh_0 in equation (61) for a translational flow is given by the equalities $Sh_0 = Sh_\beta$ (for a droplet) and $Sh_0 = Sh_\infty$ (for a solid sphere), where the quantities Sh_β and Sh_∞ are defined in equation (7); for the case of shear flow past a droplet or solid sphere it is necessary to assume in equation (61) that $Sh_0 = Sh_\beta$ or $Sh_0 = Sh_\infty$, where the parameters Sh_β and Sh_∞ are given by relations (74). It is seen from Fig. 4 that in these cases the maximum error of approximate formula (61) is about 10%.

It should be emphasized that in contrast to equation (61), the results of works [22–24, 26] cannot be presented in a simple analytical form which would be convenient for direct practical application.

Note that below a much more exact formula [than formula (61)] will be presented whose error is less than 5% [i.e. formula (96)].

8. MASS TRANSFER OF AN ARBITRARY VISCOUS DROPLET AT LARGE PECLET NUMBERS

Consider the steady-state convective diffusion to the surface of a spherical droplet in a translational Stokes flow. It is assumed that the substance, dissolved in the liquid, is being completely absorbed by the droplet surface and that its concentration far from the droplet is constant. In terms of the dimensionless variables, in the spherical coordinate system r, θ, connected with the droplet, the corresponding boundary-value problem has the form [the last boundary condition in equation (64) results from the axial symmetry of the problem; and angle θ is reckoned from the forward stagnation point on the droplet surface in the flow direction]

$$\frac{1}{\sin\theta}\left(-\frac{\partial\psi}{\partial\theta}\frac{\partial c}{\partial y}+\frac{\partial\psi}{\partial y}\frac{\partial c}{\partial\theta}\right)=\frac{1}{Pe}\frac{\partial^2 c}{\partial y^2} \tag{62}$$

$$y=0,\ c=1;\quad y\to\infty,\ c\to 0;\quad \theta=0,\ \partial c/\partial\theta=0 \tag{63}$$

$$\psi=[\lambda_1(\beta)y+\lambda_2(\beta)y^2]\sin^2\theta,$$

$$\lambda_1=\frac{1}{2(\beta+1)},\quad \lambda_2=\frac{3\beta+2}{4(\beta+1)} \tag{64}$$

$$c=\frac{C_\infty-C_*}{C_\infty},\quad y=r-1.$$

Here, instead of the exact expression for the stream function ψ corresponding to the Hadamard–Rybczinskii solution

$$\psi=\frac{1}{2}(r-1)\left[r-\frac{1}{2}\frac{\beta}{\beta+1}\left(1+\frac{1}{r}\right)\right]\sin^2\theta, \tag{65}$$

its two-term quadratic expansion in y is taken near the droplet surface, equation (64).

The solutions obtained in the diffusional boundary-layer approximation ($Pe \gg 1$) [5] correspond to the values $\lambda_1 = \lambda_1(\beta)$, $\lambda_2 = 0$ [for a moderately viscous droplet $\beta = 0(1)$] and $\lambda_1 = 0$, $\lambda_2 = 3/4$ (for a solid sphere, $\beta = \infty$). In view of the fact that $\lambda_1(\beta) \to 0$ for $\beta \to \infty$, the allowance for the second term in the stream function expansion (64) is required to obtain the sought-after relations which would be uniformly applicable with respect to the parameter β at large Peclet numbers $Pe \gg 1$. An exact analytical solution of problem (62)–(64) is nonexistent.

In the vicinity of the forward stagnation point, $\theta = 0$, by virtue of the equalities $\psi(y, 0) = 0$ and $(\partial c/\partial\theta)_{\theta=0} = 0$, the second term on the LHS of equation (62) can be neglected; then equation (62) degenerates into an ordinary differential equation, and the corresponding problem is written as

$$-2[\lambda_1(\beta)y+\lambda_2(\beta)y^2]\frac{dc}{dy}=\frac{1}{Pe}\frac{d^2c}{dy^2};$$

$$y=0,\ c=1;\quad y\to\infty,\ c\to 0. \tag{66}$$

Note that the solution of equation (66) depends weakly on the form of the parameter $\lambda_2 = \lambda_2(\beta)$. In fact, since in this case $Pe \gg 1$, then at $\beta < 0(Pe^{1/3})$ the quadratic term in equation (66) can be neglected, which corresponds to the solution which is independent of λ_2 at all. The necessity to allow for the terms including the dependence on λ_2 appears at $\beta \geqslant 0(Pe^{1/3})$, when (by virtue of $Pe \gg 1$) the relation $\lambda_2(\beta) \approx \lambda_2(\infty) = 3/4$ turns to be valid. This is also valid for the initial boundary-value problem (62)–(64) and allows the relationship $\lambda_2 = \lambda_2(\beta)$ to be replaced everywhere by the limiting value $\lambda_2(\infty) = 3/4$.

It is natural that for the model equation, corresponding to partial differential equation (62), the ordinary differential equation (66), having similar limiting properties as to the parameters β and Pe, has been chosen setting $\lambda_2(\beta) = \lambda_2(\infty)$ in it.

The solution of problem (66) has the form

$$c=\frac{\int_y^\infty \exp\left[-\frac{Pe}{2}\left(\frac{y^2}{\beta+1}+y^3\right)\right]dy}{\int_0^\infty \exp\left[-\frac{Pe}{2}\left(\frac{y^2}{\beta+1}+y^3\right)\right]dy}. \tag{67}$$

The local diffusional flux at the forward stagnation point of the droplet, corresponding to solution (67), is defined by the expression

$$j=\left\{\int_0^\infty \exp\left[-\frac{Pe}{2}\left(\frac{y^2}{\beta+1}+y^3\right)\right]dy\right\}^{-1}. \tag{68}$$

Taking into account that $Pe \gg 1$, the following asymptotic expressions can be obtained from formulae (67) and (68)

$$\beta = 0(1),\quad j\to j_\beta = (2Pe)^{1/2}[\pi(\beta+1)]^{-1/2} \tag{69}$$

$$\beta\to\infty,\quad j\to j_\infty = 3\cdot 2^{-1/3}\Gamma^{-1}(1/3)Pe^{1/3}.$$

Here j_β and j_∞ are the local diffusional fluxes at the forward stagnation point of a moderately viscous

droplet and of a solid sphere; it should be noted that

$$j_\infty \neq \lim_{\beta\to\infty} j_\beta = 0.$$

From asymptotic formulae (69), express the parameters β and Pe in terms of j_β and j_∞ and substitute these into equality (68). This results in

$$\frac{j}{j_\beta} = \frac{\sqrt{\pi}}{2}\left\{\int_0^\infty \exp\left[-z^2-\alpha\left(\frac{j_\infty}{j_\beta}\right)^3 z^3\right]dz\right\}^{-1},$$

$$\alpha = \left[\frac{2\Gamma(1/3)}{3\sqrt{\pi}}\right]^3 = 1.023. \qquad (70)$$

In order to obtain the sought-after expression for the mean Sherwood number per droplet surface, the analogy principle will be used, namely it will be assumed that the functional dependence of the mean Sherwood number Sh on the auxiliary Sherwood numbers, equation (7), which are governed by the asymptotic solutions of the initial boundary-value problem (62)–(64) at $\beta = 0(1)$ and $\beta \to \infty$ ($Pe \gg 1$), is similar to relation (70) for the local diffusional fluxes which correspond to model problem (66), i.e. the following formula is valid

$$\frac{Sh}{Sh_\beta} = \frac{\sqrt{\pi}}{2}\left\{\int_0^\infty \exp\left[-z^2-\alpha Q^3 z^3\right]dz\right\}^{-1},$$

$$Q = \frac{Sh_\infty}{Sh_\beta}. \qquad (71)$$

The approximate expression (7), (71) fairly well reflects the behaviour of the mean Sherwood number over the entire range of the parameter $\beta: 0 \leqslant \beta \leqslant \infty$. In particular, in the limiting cases when $\beta = 0(1)$ or $\beta \to \infty$ ($Pe \gg 1$), formula (71) passes over into relations (7) [5]. Moreover, it follows from the form of equation (71) that here there is the same similarity parameter Q which enters into equation (13) and which reveals the following relationship between the mean Sherwood numbers $Sh = Sh(Pe, \beta)$

$$\frac{Sh(Pe_1,\beta_1)}{Sh(Pe_2,\beta_2)} = \left[\frac{Pe_1(1+\beta_2)}{Pe_2(1+\beta_1)}\right]^{1/2} \quad \text{at} \quad \frac{Pe_1^{1/3}}{\beta_1+1} = \frac{Pe_2^{1/3}}{\beta_2+1}. \qquad (72)$$

It is important to stress that despite the fact that relations (72) were obtained from approximate formula (71), they are an exact consequence of the initial boundary-value problem (62)–(64) [at $\lambda_2 \cong \lambda_2(\infty) = 3/4$] which has the same similarity parameter Q. This can be proved by introducing the new variable $y_* = Pe^{1/2}(\beta+1)^{-1/2}y$ directly into equations (62)–(64).

In ref. [9], the boundary-layer equation was analyzed by a numerical method without stream function linearization. It should be borne in mind, however, that in order to obtain the required accuracy at large Peclet numbers it is necessary that the step of the mesh be selected and be substantially dependent on the values of the parameters β and Pe; and in conformity with what has been said in the case of numerical investigation of problem (62)–(64) for $Pe \gg 1$ and $0 \leqslant \beta < \infty$, the choice of the mesh step should be based on the value of the similarity parameter Q {this has not been done in work [9] and lead to the corresponding errors in calculations at large Peclet numbers $Pe = 5000$ when the mean diffusional boundary layer thickness in the limiting cases of $\beta = 0$ and $\beta = \infty$ differ already substantially (more than three times) (see Section 2)}; in this case formulae (72) may serve as a criterion for the accuracy of calculation.

Once again an important factor should be stressed which is typical for the use of the method of model equations and analogies. Namely, the range of validity of approximate formula (71) turns to be much wider than that information which is contained in the initial statement of problem (62)–(64). In particular, the approximate expression (71) can also be successfully used for the determination of the mean Sherwood number in the case of axisymmetric linear shear flow past a droplet over the entire range of the droplet to surrounding liquid viscosity ratio $0 \leqslant \beta \leqslant \infty$ at large Peclet numbers. The unperturbed velocity field of such a flow far from the droplet has the following form in the Cartesian coordinate system X_1, X_2, X_3

$$r_* \to \infty, \quad \mathbf{V} = \{GX_1, GX_2, -2GX_3\}, \qquad (73)$$

where G is the shear factor. To make equation (73) more readily comprehensible, the dimensional coordinates and quantities have been utilized. The solution of the hydrodynamic problem of the flow around a spherical droplet, complying with the boundary condition at infinity (73), has been obtained in the Stokes approximation in ref. [27]. In this case, when calculating the mass transfer between a droplet and a flow, it is necessary that for the parameters Sh_β and Sh_∞, figuring in approximate formula (71), the corresponding limiting values be used which were obtained in work [28]

$$Sh_\beta = (3/\pi)^{1/2}(\beta+1)^{-1/2}Pe^{1/2},$$

$$Sh_\infty = 1.22\,Pe^{1/3}, \quad Pe = a^2GD^{-1}. \qquad (74)$$

It is interesting to note that here there is the same similarity parameter Q, equation (13), and relationship (72) is valid.

It has been shown in the above example how, by using the method of model equations and analogies, it was possible to substantially improve the approximate solution of the model equation with the aid of exact asymptotic expressions (7).

Remark. It will now be shown how, using an approximate solution of model equation (66), it is possible by means of the integral method and subsequent use of the principle of analogies, to obtain formula (12) derived earlier in a different way. For this, integrate equation (66) over y from zero to infinity assuming that the quantity c and all of its derivatives tend exponentially to zero when $y \to \infty$. After some

transformations, this results in

$$2\int_0^\infty [\lambda_1(\beta)+2\lambda_2 y]c\,dy = -\frac{1}{Pe}\left(\frac{dc}{dy}\right)_{y=0}. \tag{75}$$

Here, as before, it is assumed that $\lambda_2 \approx \lambda_2(\infty) = 3/4$.

As usual, the solution of integral identity (75) is sought in the form

$$c = \varphi(y/\delta), \tag{76}$$

where the profile $\varphi = \varphi(x)$ is chosen arbitrarily and the constant δ, corresponding to the diffusional boundary-layer thickness at the forward stagnation point of the droplet, is determined by solving the equation

$$j^3 - 2Pe\lambda_1(\beta)\sigma_1 j - 4Pe\lambda_2\sigma_2 = 0, \quad j \equiv A\delta^{-1} \tag{77}$$

$$A = -\left(\frac{d\varphi}{dx}\right)_{x=0}, \quad \sigma_n = A^n\int_0^\infty x^{n-1}\varphi(x)\,dx,$$

which has been obtained after several transformations as a result of substitution of equation (76) into identity (75).

When $Pe \to \infty$, cubic equation (77) yields the asymptotic expressions

$$\beta = 0(1), \quad j \to j_\beta = [2Pe\lambda_1(\beta)\sigma_1]^{1/2}, \tag{78}$$

$$\beta \to \infty, \quad j \to j_\infty = (4Pe\lambda_2\sigma_2)^{1/3}.$$

Expressing the parameters β and Pe from relation (78) in terms of j_β and j_∞ and substituting these into equation (77) yield the following equation for the local diffusional flux

$$j^3 - j_\beta^2 j - j_\infty^3 = 0. \tag{79}$$

Now, by applying to this equation the analogy principle, approximate equation (12) is obtained for the mean Sherwood number.

It should be emphasized that the final result in the form of equation (12) [or equation (79)] does not depend on the particular choice of the concentration profile $\varphi = \varphi(x)$, equation (76). Note also, that approximate expression (12) leads to exact relationship (72) for the mean Sherwood numbers. Moreover, approximate equation (12) [or equation (71)] can also be used to calculate the mass transfer of nonspherical arbitrarily viscous droplets (e.g. ellipsoidal) at large Peclet numbers.

The approximate dependence of the normalized mean Sherwood number Sh/Sh_β on the parameter $Q = Sh_\infty/Sh_\beta$, obtained by evaluating the integral of equation (71), coincides within 1% with the root of cubic equation (12). (In Fig. 1(b) these curves merge virtually into one curve).

9. MULTICOMPONENT DIFFUSION TO A REACTING PARTICLE. THE NONISOTHERMAL CASE

The method of model equations and analogies can also be used with success in more complex problems described by a system of partial differential equations. As an example, consider a multicomponent convective diffusion to a reacting particle in a laminar liquid flow with non-isothermal chemical reaction occurring on its surface at a rate arbitrarily dependent on temperature and concentrations. It is assumed that the reacting species have sufficiently small concentrations, so that the presence of the surface chemical reaction does not influence the flow and the particle parameters. The effect of thermal and pressure diffusion, etc. if also ignored.

The dimensionless equations of convective diffusion and heat conduction and also the boundary conditions expressing the uniformity of temperature and concentrations far from the particle, the 'reaction law' and heat balance on the particle surface are

$$Pe_m(\mathbf{v}\cdot\nabla)c_m = \Delta c_m \quad (m = 1,\ldots,M) \tag{80}$$

$$Pe_T(\mathbf{v}\cdot\nabla)T = \Delta T \tag{81}$$

$$r \to \infty, \quad c_m \to 0, \quad T \to 0 \tag{82}$$

$$r = 1, \quad -\frac{\partial c_m}{\partial r} = k_{ms}f_{ms}(T, c_1, \ldots, c_M) \tag{83}$$

$$r = 1, \quad \frac{\partial T}{\partial r} = \sum_{m=1}^{M} h_m\frac{\partial c_m}{\partial r} \tag{84}$$

$$c_m = \frac{C_{m\infty} - C_{m*}}{C_{m\infty}}, \quad T = \frac{T_\infty - T_*}{T_\infty},$$

$$Pe_m = \frac{aU}{D_m}, \quad Pe_T = \frac{aU}{\chi}, \quad h_m = H_m\frac{D_m C_{m\infty}}{\lambda T_\infty},$$

$$k_{ms}f_{ms}(T, c_1, \ldots, c_M) \equiv \frac{aK_{ms}}{D_m C_{m\infty}}F_{ms}(T_*, C_{1*}, \ldots, C_{M*}).$$

Here C_{m*} and T_* are the concentrations of species and temperatures in the flow, $C_{m\infty}$ and T_∞ the unperturbed concentrations and temperature at infinity, Pe_T and Pe_m the thermal and diffusional Peclet numbers, D_m the diffusion coefficients, χ and λ the thermal diffusivity and thermal conductivity of the mixture, respectively, H_m and $K_{ms}F_{ms}$ the heat and rate of reaction of mth species; M the number of species participating in reaction.

The model problem corresponding to relations (80)–(84) is

$$(c_m)''_{\xi\xi} + Pe_m(c_m)'_\xi = 0; \quad T''_{\xi\xi} + Pe_T T'_\xi = 0 \tag{85}$$

$$\xi \to \infty, \quad c_m \to 0, \quad T \to 0 \tag{86}$$

$$\xi = 0, \quad -(c_m)'_\xi = k_{ms}f_{ms}(T, c_1, \ldots, c_M),$$

$$T'_\xi = \sum_{m=1}^{M} h_m(c_m)'_\xi. \tag{87}$$

The solution of equations (85), satisfying the boundary conditions at infinity (86), is given by the expressions

$$c_m = A_m\exp(-Pe_m\xi), \quad T = B\exp(-Pe_T\xi). \tag{88}$$

The unknown constants A_m and B are determined from boundary conditions (87)

$$A_m Pe_m = k_{ms} f_{ms}(B, A_1, \ldots, A_M),$$

$$BPe_T = \sum_{m=1}^{M} A_m Pe_m. \quad (89)$$

Now, from formulae (88) find the expressions for the local diffusional and heat fluxes

$$j_m = A_m Pe_m, \quad j_T = BPe_T. \quad (90)$$

Also, it should be taken into account that the values $A_m = B = 1$ in equations (88) correspond to a purely diffusional (thermal) mode of reaction on the 'particle surface', i.e. to the solution of equations (85) with boundary conditions at infinity (86) and with the simplest linear boundary conditions at $\xi = 0$

$$\xi = 0, \quad c_m = 1, \quad T = 1 \quad (91)$$

[the solution of problem (85), (86), (91) splits in this case into the solution of $(M+1)$ independent identical problems which differ only by the value of the parameters Pe_m and Pe_T]. Assuming in formulae (90) that $A_m = B = 1$, the following is obtained for a purely diffusional regime

$$j_{m0} = Pe_m, \quad j_{T0} = Pe_T. \quad (92)$$

Now, eliminating from expressions (89) the constants A_m, B and Pe_m, Pe_T by the use of equalities (90) and (92) yield the following equations for the local fluxes

$$j_m = k_{ms} f_{ms}\left(\frac{j_T}{j_{T0}}, \frac{j_1}{j_{10}}, \ldots, \frac{j_M}{j_{M0}}\right), \quad j_T = \sum_{m=1}^{M} h_m j_m. \quad (93)$$

From this, by virtue of the analogy principle, a system of algebraic (transcendental) equations is obtained to determine the mean Sherwood and Nusselt numbers

$$Sh_m = k_{ms} f_{ms}\left(\frac{Nu}{Nu_0}, \frac{Sh_1}{Sh_{10}}, \ldots, \frac{Sh_M}{Sh_{M0}}\right) \quad (m = 1, \ldots, M) \quad (94)$$

$$Nu = \sum_{m=1}^{M} h_m Sh_m.$$

It should be noted that in order to determine the auxiliary Sherwood, Sh_{m0}, and Nusselt, Nu_0, numbers, which figure in the system of equations (94), it is sufficient to know the solution of only one linear boundary-value problem (22), (26) in the entire range of Peclet numbers $0 \leqslant Pe < \infty$; in this case the values of Sh_{m0} and Nu_0 are calculated by formula (25) at $Pe = Pe_m$ and $Pe = Pe_T$, respectively.

It follows from the results of works [14, 16] that the system of equations (94) is asymptotically accurate at small Peclet numbers for a translational [14, 16] and shear [16] flow past a spherical particle (for any functions f_{ms}).

10. SOME REMARKS AND GENERALIZATIONS

It is important to note that the method of asymptotic correction, described in Section 2, is also very useful in those cases when there is an exact analytical result, i.e. when initial formula (1) is an exact rather than an approximate one. This is due to the fact that the final functional relation (5) [or (6)] between the initial quantity S and its asymptotic expressions (4) can be already used to describe approximately a substantially wider class of problems expressing qualitatively similar phenomena or processes which differ only by geometric characteristics, i.e. by the shape, flow structure, etc. In other words, the amount of information in final formula (6) turns generally to be much greater than in initial expression (1). Naturally, in this way one can expand the range of validity of sufficiently good approximate formulae (which themselves call for no improvement), derived by any approximate analytical or numerical methods and also empirically; as a result of the treatment of experimental data.

Illustrate the above on a particular example. In ref. [25], the relationship has been obtained, in the diffusional boundary-layer approximation, between the mean Sherwood number and time per spherical bubble in a developed axisymmetric linear shear flow

$$Sh = (3Pe/\pi)^{1/2} cth^{1/2}(3Pet), \quad Pe = a^2 G D^{-1} \quad (95)$$

{the corresponding nonstationary boundary-value problem is formulated by equations, initial and boundary conditions (56), (57) with allowance for the fact that liquid velocity distribution outside the bubble is governed as in ref. [27]}.

Formula (95) can be restated in a more convenient form by using the asymptotic value of the mean Sherwood number $Sh_0 = (3Pe/\pi)^{1/2}$ which corresponds to the developed process of diffusion for $t \to \infty$. Namely, the elimination of the Peclet number Pe through the use of Sh_0 in formula (95) gives

$$\frac{Sh}{Sh_0} = cth^{1/2}(\pi Sh_0^2 t). \quad (96)$$

If now, one will 'forget' that the coefficient Sh_0 in equation (96) is specified by the asymptotic expression formula (95) for $t \to \infty$ and will determine it directly from the solution of the corresponding stationary problem of convective mass and heat transfer at large Peclet numbers, then formula (96) can already be successfully used for an approximate description of the unsteady-state processes of diffusion to the surfaces of droplets, particles and bubbles. In particular, a direct check can easily show that the substitution of $Sh_0 = Sh_\beta$ into equation (96), where Sh_β is defined by equation (74), leads to a correct result obtained in the boundary-layer approximation for the unsteady-state mass transfer between a moderately viscous droplet ($0 \leqslant \beta \leqslant 0(1)$) and a flow. Moreover, the comparison

between the approximate formula obtained by the substitution of $Sh_0 = Sh_\beta$ into equation (96), where Sh_β is specified by equation (7) and corresponds to mass transfer between a spherical droplet and a translational Stokes flow, with the results of refs. [22–24] shows that the maximum error of formula (96) in this case amounts only to 1%; the respective curves in Fig. 4 do not virtually differ {recall that the results of refs. [22–24] for the mean Sherwood number are given in the form of a sufficiently complex integral which cannot be given in a simple analytical form of the type of equation (96)}. For a solid sphere, formula (96) differs from the results of ref. [26] by 4%.

It also follows from the foregoing that, to retain generality, it is much more preferable that the final results of any analytical, numerical and experimental investigation be treated in 'asymptotic coordinates' of the type used earlier in this paper (see the way the axes are denoted in Figs 1(b) and 2–4) than the initial and sought-after dimensionless parameters of the problem (e.g. Pe, k_s, k_v, β, Sh, etc.). This is due to the fact that the corresponding relations, obtained for a certain particular problem and written down in 'asymptotic coordinates' can be directly used for the calculation of a number of other analogous problems.

The procedure of asymptotic correction allows one, by using very simple means, literally the 'handwaving', (by improving the results of any reasonable approximate method applied in the study of problems, phenomena or processes), to obtain the sought-after approximate relationships. It is evident that in constructing initial approximate formula (1) it is advisable to start with the analysis of simplest particular problems corresponding to idealized (model) phenomena or processes occurring in simple conditions in the fields of simplest geometry. These very reasonings served as the basis for the formulation of the method of model equations and analogies suggesting the result of the solution of much more simple model auxiliary problem to be used as initial approximate relation (1).

The comparison of the results obtained with a number of particular characteristic cases, for whose verification approximate or asymptotic formulae are already available, shows a good accuracy and wide possibilities of the methods suggested. Besides the mass and heat transfer theory, these methods can also be used for the investigation of the problems of chemical technology, chemical engineering, macrokinetics, hydrodynamics and other fields of science and technology, where there is a necessity for constructing different kinds of approximate (engineering) formulae.

In a number of cases the analogy principle, understood in a wider sense than in Section 3, can presumably be successfully used also as a basis for the construction of approximate formulae for very differing physical phenomena (which are described by different equations) by establishing qualitatively analogous characteristics. This can be illustrated on some specific examples.

It follows from the results of refs. [29, 30] that at small and moderate Reynolds numbers ($0 \leqslant Re \leqslant 50$) the resistance coefficient of a spherical droplet in a translational flow of a viscous incompressible liquid is well described by the following approximate relation

$$C_x(\beta, Re) = \frac{\beta}{\beta+1} C_x(\infty, Re) + \frac{1}{\beta+1} C_x(0, Re). \quad (97)$$

Using this formula, at the prescribed resistance coefficients of a solid sphere $C_x(\infty, Re)$ and of a gas bubble $C_x(0, Re)$, it is possible to determine the resistance coefficient of an arbitrarily viscous droplet.

In the processes of convective heat and mass transfer, the Peclet number is an analogue of the Reynolds number, and the mean Nusselt (Sherwood) number is an analogue of the resistance coefficient. Therefore, it would be reasonable to suppose that in the problem of heat and mass transfer between a spherical arbitrarily viscous droplet and a translational flow at small and moderate Peclet numbers, the following approximate relation, similar to relation (97), would be valid

$$Sh(\beta, Pe) = \frac{\beta}{\beta+1} Sh(\infty, Pe) + \frac{1}{\beta+1} Sh(0, Pe), \quad (98)$$

which is obtained from relation (97) after the replacement $C_x \to Sh$, $Re \to Pe$.

A direct comparison with the available asymptotic and numerical results shows [17, 30] that approximate formula (96), obtained on such simple grounds, turns to be quite good for the practical use within the range $0 \leqslant Pe \leqslant 500$.

Consider now the convective mass transfer to a body of revolution of any shape arbitrarily oriented in a translational flow of viscous incompressible liquid. In ref. [30], for the mean Sherwood number the following expression has been obtained

$$Sh = Sh_{\parallel} \cos^2 \omega + Sh_{\perp} \sin^2 \omega, \quad (99)$$

where ω is the angle between the axis of the body of revolution and the direction of the incident flow; $Sh_{\parallel}$ and $Sh_{\perp}$ are the mean Sherwood numbers corresponding to the body of revolution located parallelly and perpendicularly in the flow. Formula (99) is an asymptotically correct one at small Peclet numbers (accurate to the terms of the order of $Pe^2 \ln Pe$ including) and provides the possibility to approximately determine, from the available values $Sh_{\parallel}$ and $Sh_{\perp}$, the mean Sherwood number for any orientation of the body of revolution in the translational flow at small and moderate Peclet numbers.

Formula (99) can be used as a basis for the construction of the respective approximate relation in the problem of the hydrodynamic flow past a body of revolution arbitrarily oriented in a translational flow of a viscous incompressible liquid. Namely, the analogue of the mean Sherwood number in the hydrodynamic problem is the resistance coefficient of the body of revolution $C_x = (\mathbf{F} \cdot \mathbf{i})(\frac{1}{2}\rho U_\infty^2 \Sigma_*)^{-1}$ ($\mathbf{F}$ is the total

resistance force of the body, $\mathbf{i}$ is the unit directing vector of the liquid velocity at infinity, ρ the liquid density, U_∞ the unperturbed flow velocity at infinity, Σ_* the dimensional surface area of the body), which has been determined from the body resistance force projection in the free stream flow direction ($\mathbf{F} \cdot \mathbf{i}$). Taking this into account, the replacement in equation (99) of Sh by C_* leads to the following expression

$$C_x = C_{x\parallel} \cos^2 \omega + C_{x\perp} \sin^2 \omega, \qquad (100)$$

where $C_{x\parallel}$ and $C_{x\perp}$ are the resistance coefficients corresponding to a parallel ($\omega = 0$) and perpendicular ($\omega = \pi/2$) position of the body of revolution in the flow.

In the Stokes approximation (at $Re = 0$), formula (100) is an exact one for arbitrarily shaped particles; moreover, for a spherical particle, relation (100) holds identically for any Reynolds numbers. It is therefore to be expected that approximate relation (100) will give good results not only at small but at moderate Reynolds numbers numbers also (at leat for the particles whose shape does not markedly differ from a spherical one).

This part of the paper has presented the binary analytical methods which are meant for the derivation or improvement of different kinds of approximate formulae and which virtually constitute the synthesis of two successively applied methods of different type: first, any approximate (engineering) method is used whose final result is then 'improved' with the aid of familiar asymptotic expressions obtained by any modified version of the perturbation method. The method of asymptotic correction is here a prerequisite link providing this synthesis of the engineering and asymptotic methods.

The second part of the paper [30] will concern itself with the description of approximate methods fully based on the use of only asymptotic information. It will show the technique for a substantial extension of the validity range of one-sided asymptotic formulae (the method of asymptotic extrapolation), and also the technique of constructing approximate formulae on the basis of familiar two-sided asymptotic expressions (the method of asymptotic interpolation). These methods provide the possibility for the use of the results of asymptotic investigations directly in engineering practice.

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NOUVELLES METHODES DANS LA THEORIE DU TRANSFERT DE CHALEUR ET DE MASSE. I—LA METHODE DE LA CORRECTION ASYMPTOTIQUE ET LA METHODE DES EQUATIONS MODELES ET ANALOGIES

Résumé—Deux nouvelles méthodes analytiques approchées sont présentées pour étudier les problèmes de transfert de chaleur et de masse: la méthode de la correction asymptotique et la méthode des équations modèles et analogies. Des exemples spécifiques sont considérés pour illustrer l'utilisation de ces méthodes. De nouvelles formules sont obtenues et elles sont intéressantes dans la théorie du transfert de chaleur et de masse. La comparaison des résultats obtenus pour une variété de cas typiques avec des solutions exactes, approachées ou asymptotiques actuellement disponibles, montre une bonne précision et une large applicabilité des méthodes proposées. Celles-ci peuvent aussi être utilisées pour obtenir des formules approchées (ingénièrie) dans les problèmes de génie chimique, mécanique chimique, microcinétique, hydrodynamique, etc....

NEUE VERFAHREN IN DER THEORIE DES WÄRME- UND STOFFAUSTAUSCHES—I. DAS VERFAHREN DER ASYMPTOTISCHEN KORREKTUR UND DAS VERFAHREN DER MODELLGLEICHUNGEN UND ANALOGIEN

Zusammenfassung—Zwei neue analytische Näherungsverfahren zur Untersuchung von Wärme- und Stoffübergangsvorgängen werden vorgeschlagen: das Verfahren der asymptotischen Korrektur und das Verfahren der Modellgleichungen und Analogien. Einige spezielle Beispiele werden betrachtet, um die Anwendung der genannten Verfahren zu veranschaulichen. Neue Formeln wurden hergeleitet, die in der Theorie des konvektiven Stoff- und Wärmeübergangs um ihrer selbst willen interessant sind. Der Vergleich der Ergebnisse aus einer Vielzahl typischer spezieller Anwendungsfälle, für deren Berechnung exakte Formeln, Näherungsgleichungen oder asymptotische Gleichungen bereits vorlagen, zeigt eine gute Genauigkeit und weite Anwendungsmöglichkeiten der vorgeschlagenen Verfahren. Ebenso können diese Verfahren mit Erfolg angewandt werden, um Näherungsgleichungen für Probleme der chemischen und mechanischen Verfahrenstechnik, der Mikrokinetik, der Strömungsmechanik etc. aufzustellen.

НОВЫЕ МЕТОДЫ ТЕОРИИ МАССО-И ТЕПЛОПЕРЕНОСА—I. МЕТОД АСИМПТОТИЧЕСКОЙ КОРРЕКЦИИ И МЕТОД МОДЕЛЬНЫХ УРАВНЕНИЙ И АНАЛОГИЙ

Аннотация—В работе предлагаются два новых приближенных аналитических метода исследования задач массо-и теплопереноса: метод асимптотической коррекции и метод модельных уравнений и аналогий. Рассмотрен ряд конкретных примеров, иллюстрирующих использование предложенных методов. Получены новые формулы, представляющие самостоятельный интерес в теории конвективного массо-и теплопереноса. Сопоставление полученных результатов с целым рядом частных характерных случаев, для которых уже имеются необходимые для проверки точные, приближенные или асимптотические формулы, показывает хорошую точность и широкие возможности предложенных методов. Эти методы могут быть успешно использованы также для построения приближенных (инженерных) формул в задачах химической технологии, химической механики, макрокинетики, гидродинамики и др.